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The dynamic behaviour of piezoelectric laminated bars☆

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Abstract

A theory of laminated electroelastic bars with layers arranged symmetrically about the middle plane of the bar is constructed. Particular attention is given to the influence of the electrical conditions on the faces of the piezoelectric layers on the equations of the theory of bars. Formulae are obtained which, after solving the problem of a laminated bar, enable one to transfer from one-dimensional required quantities to three-dimensional required quantities. As an example, the vibrations of a three-layer electroelastic bar are considered, the displacements, stresses and electrical quantities are calculated, and the dependence of the electromechanical coupling coefficient on the frequency of the vibrations and the thicknesses of the elastic and piezoelectric layers is investigated.

Piezoelectric laminated bars have been considered in many papers (see Refs 1–5), among which is the review Ref 5. However the problems of choosing correct hypotheses for constructing a theory of bars and methods of calculating the efficiency of bars as energy converters remain to be solved.

1. Formulation of the problem

A laminated bar with elastic and piezoelectric layers placed symmetrically about the middle plane of the bar is considered. The piezoelectric layers can be made of piezoceramics or piezofilm. A cross section of the bar in Cartesian coordinates and the electrical load are presented schematically in Fig. 1.

The bar of the length l consists of $2N$ layers if the middle plane coincides with the contact plane of the layers, and $2N-1$ layers when the middle plane coincides with the middle plane of one of the layers, which will be called the central layer. In the latter case, to obtain $2N$ layers we partition the central layer into two layers, the upper layer having the number 1, and the lower one the number -1. The layers are numbered from the middle plane of the bar to the upper face from 1 to N and from the middle plane of the bar to the lower face from -1 to $-N$; the thickness of the layers with numbers k and $-k$ is h_k. The bar is referred to Cartesian coordinates; the x_1 axis is directed along the length of the bar, the x_2 axis is directed along the width and the x_3 axis is orthogonal to them.

It is assumed that the piezoelectric layers are prepolarized in the x_3 direction. As in the theory of elastic bars, the stresses σ_{22} and σ_{33} in the constitutive relations for piezoelectric bars can be neglected compared with the stress σ_{11}, and we can assume that the electroelastic state does not depend on the x_2 coordinate.

The equations for the elastic and electroelastic layers can be written, in view of the above assumptions, as follows:

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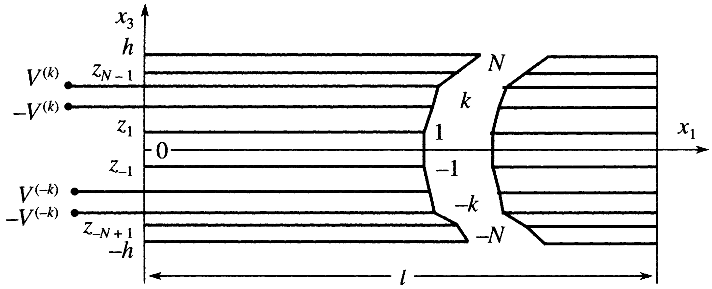

Fig. 1.

the equations of motion

$$\partial\sigma_{ii}^{(k)}/\partial x_i + \partial\sigma_{ij}^{(k)}/\partial x_j = \rho\partial^2 u_i^{(k)}/\partial t^2, \quad i \neq j = 1,3 \tag{1.1}$$

the stress - strain formulae

$$e_1^{(k)} = \partial u_1^{(k)}/\partial x_1 \tag{1.2}$$

constitutive relations for the elastic layers (Hooke's law)

$$\sigma_{11}^{(k)} = E^{(k)} e_1^{(k)} \tag{1.3}$$

and the constitutive relations for piezoelectric layers

$$\sigma_{11}^{(k)} = (e_1^{(k)} - d_{31}^{(k)} E_3^{(k)})/s_{11}^{E^{(k)}} \tag{1.4}$$

$$D_3^{(k)} = \varepsilon_{33}^{T^{(k)}} E_3^{(k)} + d_{31}^{(k)} \sigma_{11}^{(k)} \tag{1.5}$$

where

$$E_3^{(k)} = -\partial\varphi^{(k)}/\partial x_3 \tag{1.6}$$

In formulae (1.1)–(1.6) u_1 and e_1 are the displacement and strain in the x_1 direction, E_3 and D_3 are the components of the electrical field vector and electrical induction vector in the x_3 direction, s_{11}^E is the elastic compliance for zero electric field, d_{31} is the piezoelectric constant, ε_{33}^T is the permittivity for zero stresses, and φ is the electric potential. The notation used is the same as that employed previously.[6] The superscript in parentheses denotes the number of the layer.

The mechanical surface load on the faces of the bar are specified in the usual way

$$\sigma_{13}^{(\pm N)}\big|_{x_3 = \pm h} = \pm q_1^{\pm}, \quad \sigma_{33}^{(\pm N)}\big|_{x_3 = \pm h} = \pm q_3^{\pm} \tag{1.7}$$

If there are no electrodes on the surfaces of the bar and the layer on these surfaces is in contact with a non-conducting medium (say, either insulating glue or a vacuum or air), the component of the electric induction vector D_3 normal to these surfaces equals zero:

$$D_3^{(k)} = 0 \tag{1.8}$$

If the surfaces of the electroelastic layer are covered by electrodes and the electric potential on the electrodes is specified, the boundary conditions on the electrode-covered surfaces have the form

$$\begin{aligned} &\varphi^{(k)}\big|_{x_3 = z_k} = V^{(k)}, \quad \varphi^{(k)}\big|_{x_3 = z_{k-1}} = -V^{(k)} \\ &\varphi^{(-k)}\big|_{x_3 = z_{-k}} = -V^{(-k)}, \quad \varphi^{(-k)}\big|_{x_3 = z_{-(k-1)}} = V^{(-k)} \end{aligned} \tag{1.9}$$

On the short-circuit electrodes the electric potential is equal to zero

$$\varphi^{(k)}\big|_{x_3 = z_k} = \varphi^{(k)}\big|_{x_3 = z_{k-1}} = 0 \tag{1.10}$$

If the electrodes are closed by an electric circuit with known complex admittance $Y = Y_0 + iY_1$, then

$$I = \int_\Omega \frac{dD_3}{dt} d\Omega = 2VY \tag{1.11}$$

On open-circuit electrodes the following integral condition is satisfied

$$I = \int_\Omega \frac{dD_3}{dt} d\Omega = 0 \tag{1.12}$$

where I is the magnitude of current.

2. The construction of a theory of bars

In order to construct a theory of piezoelectric bars some assumptions regarding the electrical quantities must be made. As when constructing a theory of piezoelectric shells and plates,[6] the content of the accepted hypotheses depends on the electrical conditions on the faces of the piezoelectric layers.

For piezoelectric layers we will make the following assumptions, which were justified previously by an asymptotic method for single-layer electroelastic plates and shells.[6]

1°. For a piezoelectric layer with the electrode-covered faces the component D_3 of the electric induction vector normal to the surfaces does not depend on the thickness coordinate x_3

$$D_3^{(k)} = D_{3,0}^{(k)}(x_1) \tag{2.1}$$

2°. The electric potential φ both for the layer with electrodes on the faces and for the layer without electrodes is a quadratic function of the thickness coordinate x_3

$$\varphi^{(k)} = \varphi_{,0}^{(k)} + x_3\varphi_{,1}^{(k)} + x_3^2\varphi_{,2}^{(k)} \tag{2.2}$$

3°. For the electroelastic layer without electrodes the following strong inequality is satisfied

$$D_3^{(k)} \ll (\varepsilon_{33}^E E_3^{(k)}, d_{31}\sigma_{11}^{(k)})$$

In line with Assumption 3°, Eq. (1.5) for the layer without electrodes can be rewritten in the form

$$\varepsilon_{33}^{T^{(k)}} E_3^{(k)} + d_{31}^{(k)}\sigma_{11}^{(k)} = 0 \tag{2.3}$$

If the electric potential on the electrodes is given by (1.9), formula (2.2) can be transformed to the form

$$\varphi^{(\pm k)} = \mp V^{(\pm k)} + (x_3 - z_{\pm(k-1)})(2V^{(\pm k)}/h_k \mp h_k\varphi_{,2}^{(\pm k)}) + (x_3 - z_{\pm(k-1)})^2\varphi_{,2}^{(\pm k)} \tag{2.4}$$

Here and henceforth the superscript (k) k takes the values $-N, \ldots, -2, -1, 1, 2, \ldots, N$, while the superscript k with a double sign $(\pm k)$ takes only positive integer values $1, 2, \ldots, N$.

Taking relations (2.4) and (1.6) into account, we obtain

$$E_{3,0}^{(\pm k)} = -2V^{(\pm k)}/h_k + (z_{\pm k} + z_{\pm(k-1)})\varphi_{,2}^{(\pm k)}$$
$$E_{3,1}^{(k)} = E_{3,1}^{(-k)} = -2\varphi_{,2}^{(k)} = -2\varphi_{,2}^{(-k)}, \quad \varphi_{,2}^{(k)} = \frac{d_{31}^{(k)}}{2\varepsilon_{33}^{T^{(k)}}}\sigma_{11,1}^{(k)} = \frac{d_{31}^{(k)}}{2\varepsilon_{33}^{T^{(k)}}s^{(k)}}\kappa \qquad (2.5)$$

where

$$E_3^{(k)} = E_{3,0}^{(k)} + x_3E_{3,1}^{(k)}, \quad s^{(k)} = s_{11}^{E^{(k)}}(1-(k_{31}^{(k)})^2) \qquad (2.6)$$

For the mechanical quantities of any layer Kirchhoff's hypotheses hold, and they can therefore be written in the form of the following linear functions of x_3

$$u_1 = u_{1,0} + x_3u_{1,1}, \quad u_3 = u_{3,0}, \quad e_1 = \varepsilon + x_3\kappa, \quad \sigma_{11}^{(k)} = \sigma_{11,0}^{(k)} + x_3\sigma_{11,1}^{(k)} \qquad (2.7)$$

$$\varepsilon = \partial u/\partial x_1, \quad \kappa = \partial^2 w/\partial x_1^2, \quad u = u_1|_{x_3=0}, \quad w = -u_3|_{x_3=0}, \quad u_{1,1} = -\partial w/\partial x_1 \qquad (2.8)$$

Here ε and κ are the components of the tangential and bending strains of the middle line of the bar, u and w are the tangential displacement and deflection of the points of the middle line respectively, and $u_{1,1}$ is the angle of rotation of an element normal to the middle line.

Taking into account formulae (2.5)–(2.8), the constitutive relations for the k-th piezoelectric layer with electrodes on the faces can be rewritten in the form

$$\sigma_{11,0}^{(k)} = \frac{1}{s_{11}^{E^{(k)}}}\varepsilon + \frac{d_{31}^{(k)}}{s_{11}^{E^{(k)}}}\frac{2V^{(k)}}{h_k} - \frac{(z_k + z_{k-1})(k_{31}^{(k)})^2}{s^{(k)}}\kappa, \quad \sigma_{11,1}^{(k)} = \frac{1}{s^{(k)}}\kappa \qquad (2.9)$$

Similarly, the constitutive relations (1.4) for the k-th piezoelectric layer without electrodes on the faces, taking formulae (2.3) and (2.7) into account, can be written as follows:

$$\sigma_{11,0}^{(k)} = \frac{1}{s^{(k)}}\varepsilon, \quad \sigma_{11,1}^{(k)} = \frac{1}{s^{(k)}}\kappa \qquad (2.10)$$

and the relations of elasticity for the elastic layer finally take the usual form

$$\sigma_{11,0}^{(k)} = E^{(k)}\varepsilon, \quad \sigma_{11,1}^{(k)} = E^{(k)}\kappa \qquad (2.11)$$

Integrating the stresses over the thickness we obtain the resultant tangential force T and bending moment G

$$T = -\int_{-h}^{+h}\sigma_{11}dx_3 = \sum_{k=1}^{N}\left(\int_{z_{k-1}}^{z_k}\sigma_{11}^{(k)}dx_3 + \int_{z_{-k}}^{z_{-k+1}}\sigma_{11}^{(-k)}dx_3\right)$$
$$G = -\int_{-h}^{+h}\sigma_{11}x_3dx_3) = -\sum_{k=1}^{N}\left(\int_{z_{k-1}}^{z_k}\sigma_{11}^{(k)}x_3dx_3 + \int_{z_{-k}}^{z_{-k+1}}\sigma_{11}^{(-k)}x_3dx_3\right) \qquad (2.12)$$

After rearrangement, we obtain the following constitutive relations for the theory of laminated electroelastic bars

$$T = A\varepsilon + P, \quad G = M\kappa + Q \qquad (2.13)$$

Table 1

Notation	Elastic layer	Piezoelectric layer without electrodes	Piezoelectric layer with electrodes
a_k	$E^{(k)}$	$1/s^{(k)}$	$1/s_{11}^{E(k)}$
m_k	$E^{(k)}$	$1/s^{(k)}$	$1/s_{11}^{E(k)} + h_k^3 (k_{31}^{(k)})^2 / [4(z_k^3 - z_{k-1}^3) s^{(k)}]$
p_k	0	0	$d_{31}^{(k)}/s_{11}^{E(k)}$

Here

$$A = 2\sum_{k=1}^{N} h_k a_k, \quad M = -\frac{2}{3}\sum_{k=1}^{N}(z_k^3 - z_{k-1}^3)m_k$$
$$P = 2\sum_{k=1}^{N} p_k(V^{(k)} + V^{(-k)}), \quad Q = -\sum_{k=1}^{N}\frac{z_k^2 - z_{k-1}^2}{h_k}p_k(V^{(k)} - V^{(-k)}) \tag{2.14}$$

The notation employed is shown in Table 1.

The values of the electric potential occur in the formulae defining the quantities P and Q. They are either given or are found from integral condition (1.11) for an electric circuit with known complex admittance and conditions (1.12) in the case of open-circuit electrodes.

The equilibrium equations in the theory of laminated electroelastic bars of symmetrical structure have exactly the same form as in the case of elastic bars

$$\frac{\partial T}{\partial x_1} + X = 2h\rho\frac{\partial^2 u}{\partial t^2}, \quad N = \frac{\partial G}{\partial x_1}, \quad \frac{\partial N}{\partial x_1} + Z = 2h\rho\frac{\partial^2 w}{\partial t^2}, \quad N = -\int_{-h}^{+h}\sigma_{13}dx_3$$
$$\rho = \frac{1}{h}\sum_{k=1}^{n}\rho_k h_k, \quad X = q_1^+ + q_1^-, \quad Z = -(q_3^+ + q_3^-) \tag{2.15}$$

Here N is the shearing force.

The problem under consideration, as well as in the theory of elastic bars, is divided into two problems: the plane problem

$$T = A\varepsilon + P, \quad \frac{\partial T}{\partial x_1} + X = 2h\rho\frac{\partial^2 u}{\partial t^2}, \quad \varepsilon = \frac{\partial u}{\partial x_1} \tag{2.16}$$

and the problem of the bending of a bar

$$G = M\kappa + Q, \quad N = \frac{\partial G}{\partial x_1}, \quad \frac{\partial N}{\partial x_1} + Z = 2h\rho\frac{\partial^2 w}{\partial t^2}, \quad \lambda = \frac{\partial^2 w}{\partial x_1^2} \tag{2.17}$$

Here A, M, P, Q, ρ are the constants defined above. To solve both problems, since they are identical with the corresponding problems of the theory of elastic bars, apart from constant coefficients, we will use the well-developed methods of this theory.

Let us analyse the hypotheses that were used to reduce the three-dimensional problem to a one-dimensional problem of the theory of bars. It should be emphasized that for piezoelectric layers with electrodes and without them different theories are used, and as a result different coefficients are obtained in the one-dimensional relations of electroelasticity (2.13), (2.14). Besides, many authors erroneously assume a linear variation of the electric potential over the thickness of the bar. However, for a linear variation the quantity E_3 is constant

$$E_3^{(k)} = E_{3,0}^{(k)} = -2V^{(k)}/h_k \tag{2.18}$$

and the formulae for the stresses of the piezoelectric layer take the form

$$\sigma_{11,0}^{(k)} = \frac{1}{s_{11}^{E^{(k)}}}\varepsilon + \frac{2V^{(k)}}{h_k}\frac{d_{31}^{(k)}}{s_{11}^{E^{(k)}}}, \quad \sigma_{11,1}^{(k)} = \frac{1}{s^{(k)}}\kappa \tag{2.19}$$

Substituting relations (2.18) and (2.19) into Eq. (1.5), taking into account the fact that $D_3^{(k)} = D_{3,0}^{(k)}(x_1)$ and equating the coefficients of like powers of x_3, we obtain the equalities

$$D_{3,0}^{(k)} = (\varepsilon_{33}^{T})^{(k)}E_{3,0}^{(k)} + d_{31}^{(k)}\sigma_{11,0}^{(k)}, \quad \sigma_{11,1}^{(k)} = 0$$

the second of which for the bending problem makes no sense. This means that the hypothesis that the electric potential varies linearly, in general, is incorrect.

3. The change from one-dimensional quantities of the theory of bars to three-dimensional quantities

After solving the problem for an electroelastic bar we must transfer from the one-dimensional required quantities obtained to three-dimensional quantities – displacements, stresses and electrical quantities.

We recall that in the plane problem σ_{11}, σ_{33}, ε, E_3 and D_3 are even functions of x_3, whereas in the bending problem the quantities σ_{13} and u are odd functions.

In line with the above, we will represent the mechanical and electrical loads as the sum of even and odd functions of x_3. The even part of the surface load σ_{33} and the odd part of the surface load σ_{13} must be taken into account when solving the plane problem, while the odd part of the surface load σ_{33}, the even part of the surface load σ_{13} and the electric potential φ must be taken into account when solving the bending problem.

3.1. The plane problem

The conditions on the faces of the bar, taking into account the evenness and oddness, can be written in the form

$$\begin{aligned} &\sigma_{13}^{(\pm N)}\big|_{x_3 = \pm h} = \pm\frac{1}{2}(q_1^+ + q_1^-), \quad \sigma_{33}^{(\pm N)}\big|_{x_3 = \pm h} = \frac{1}{2}(q_3^+ - q_3^-) \\ &\varphi^{(\pm k)}\big|_{x_3 = z_{\pm k}} = \pm\frac{1}{2}(V^{(k)} + V^{(-k)}), \quad \varphi^{(\pm k)}\big|_{x_3 = z_{\pm(k-1)}} = \mp\frac{1}{2}(V^{(k)} + V^{(-k)}) \end{aligned} \tag{3.1}$$

Formulae for the three-dimensional quantities in the region $x_3 \geq 0$ are written below; formulae for $x_3 \leq 0$ can be written by analogy, taking into account the evenness or oddness of the appropriate quantities relative to the middle plane of the bar.

In the case of the plane problem the electric potential of each piezoelectric layer is a linear function of x_3 ($\varphi_{,2} = 0$, since in the plane problem $\kappa = 0$), and $E_3^{(k)}$ does not depend on x_3:

$$\varphi^{(k)} = \frac{1}{h_k}\left(x_3 - z_k + \frac{h_k}{2}\right)(V^{(k)} + V^{(-k)}), \quad E_3^{(k)} = E_{3,0}^{(k)} = -\frac{1}{h_k}(V^{(k)} + V^{(-k)}) \tag{3.2}$$

The displacement u and the strain ε are independent of the x_3 coordinate, they are the same for any layer and are found by solving the one-dimensional problem. Hence the stress σ_{11} and the quantity D_3 for any piezoelectric layer are also independent of x_3 and are given by the formulae

$$\sigma_{11,0}^{(k)} = \frac{1}{s_{11}^{E^{(k)}}}\varepsilon - \frac{d_{31}^{(k)}}{s_{11}^{E^{(k)}}}E_{3,0}^{(k)}, \quad D_{3,0}^{(k)} = \varepsilon_{33}^{T^{(k)}}E_{3,0}^{(k)} + d_{31}^{(k)}\sigma_{11,0}^{(k)} \tag{3.3}$$

If the faces of the electroelastic layer have no electrodes, we can obtain from formulae (3.3) by virtue of Eq. (2.3)

$$E_{3,0}^{(k)} = -\frac{d_{31}^{(k)}}{\varepsilon_{33}^{T^{(k)}}}\sigma_{11,0}^{(k)}, \quad \sigma_{11,0}^{(k)} = \frac{1}{s^{(k)}}\varepsilon \tag{3.4}$$

For the stresses of the elastic layer the usual formula holds, i.e.

$$\sigma_{11,0}^{(k)} = E^{(k)}\varepsilon \tag{3.5}$$

In the plane problem we take for the stresses σ_{13} and σ_{33} the following laws of the variation over the thickness of the bar

$$\sigma_{13} = x_3\sigma_{13,1}, \quad \sigma_{33} = \sigma_{33,0} + x_3^2\sigma_{33,2} \tag{3.6}$$

We will find the approximate values of the stresses σ_{13} and σ_{33}, which satisfy conditions (3.1) on the faces of the bar and the equations of motion. We obtain

$$\sigma_{13} = \frac{x_3}{2h}(q_1^+ + q_1^-), \quad \sigma_{33} = \frac{1}{2}(q_3^+ - q_3^-) + \frac{h^2 - x_3^2}{2h}\frac{\partial}{\partial x_1}(q_1^+ + q_1^-) \tag{3.7}$$

Formulae (3.7) define the stresses as continuous functions, and hence the conditions for the stresses on the contact surface of the layers to be equal are satisfied automatically.

If necessary, the stresses σ_{13} and σ_{33} can be determined in a higher approximation. To do this the equations of motion (1.1) must be integrated for each layer separately

$$\frac{\partial\sigma_{13}^{(k)}}{\partial x_3} = -\frac{\partial\sigma_{11}^{(k)}}{\partial x_1} + \rho_k\frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial\sigma_{33}^{(k)}}{\partial x_3} = -\frac{\partial\sigma_{31}^{(k)}}{\partial x_1} - \rho_k\frac{\partial^2 w}{\partial t^2} \tag{3.8}$$

and should satisfy conditions (3.1) on the faces of the bar and the conditions for the stresses σ_{33} and σ_{31} to be equal on the contact surfaces of the layers. As a result we obtain

$$\begin{aligned}
\sigma_{31}^{(N)} &= \left(-\frac{\partial\sigma_{11,0}^{(N)}}{\partial x_1} + \rho_N\frac{\partial^2 u}{\partial t^2}\right)(x_3 - h) + \frac{1}{2}(q_1^+ + q_1^-)\\
\sigma_{31}^{(k-1)} &= \left(-\frac{\partial\sigma_{11,0}^{(k-1)}}{\partial x_1} + \rho_{k-1}\frac{\partial^2 u}{\partial t^2}\right)(x_3 - z_{k-1}) + \sigma_{31}^{(k)}\big|_{x_3 = z_{k-1}}, \quad k = N, N-1, \dots, 3\\
\sigma_{31}^{(1)} &= \frac{x_3}{h_1}\sigma_{13}^{(2)}\big|_{x_3 = h_1}\\
\sigma_{33}^{(N)} &= \frac{1}{2}\left(\frac{\partial^2\sigma_{11,0}^{(N)}}{\partial x_1^2} - \rho_N\frac{\partial^3 u}{\partial t^2\partial x_1}\right)(x_3^2 - 2hx_3 + h^2) - \frac{1}{2}\frac{\partial}{\partial x_1}(q_1^+ + q_1^-)(x_3 - h) + \frac{1}{2}(q_3^+ - q_3^-)\\
\sigma_{33}^{(k-1)} &= \left(\frac{\partial^2\sigma_{11,0}^{(k-1)}}{\partial x_1^2} - \rho_{k-1}\frac{\partial^3 u}{\partial t^2\partial x_1}\right)(x_3^2 - 2z_{k-1}x_3 + z_{k-1}^2) + \sigma_{33}^{(k)}\big|_{x_3 = z_{k-1}}, \quad k = N, N-1, \dots, 2
\end{aligned} \tag{3.9}$$

3.2. The bending problem

The conditions on the faces of the bar and on the electrode-covered surfaces have the form

$$\begin{aligned}
&\sigma_{31}^{(\pm N)}\big|_{x_3 = \pm h} = \frac{1}{2}(q_1^+ - q_1^-), \quad \sigma_{33}^{(\pm N)}\big|_{x_3 = \pm h} = \pm\frac{1}{2}(q_3^+ + q_3^-)\\
&\varphi^{(\pm k)}\big|_{x_3 = z_{\pm k}} = \frac{1}{2}(V^{(k)} - V^{(-k)}), \quad \varphi^{(\pm k)}\big|_{x_3 = z_{\pm(k-1)}} = -\frac{1}{2}(V^{(k)} - V^{(-k)})
\end{aligned} \tag{3.10}$$

We obtain for the piezoelectric layer with electrodes on the faces with a given electric potential

$$\begin{aligned}
\varphi^{(k)} &= -\frac{1}{2}(V^{(k)} - V^{(-k)}) + (x_3 - z_{(k-1)})\left(\frac{V^{(k)} - V^{(-k)}}{h_k} - h_k\varphi_{,2}^{(k)}\right) + (x_3 - z_{(k-1)})^2\varphi_{,2}^{(k)} \\
E_{3,0}^{(k)} &= -\frac{V^{(k)} - V^{(-k)}}{h_k} + (z_k + z_{k-1})\varphi_{,2}^{(k)}, \quad E_{3,1}^{(k)} = -2\varphi_{,2}^{(k)} = -\frac{d_{31}^{(k)}}{\varepsilon_{33}^{T^{(k)}}}\sigma_{11,1}^{(k)} \\
\sigma_{11,0}^{(k)} &= -\frac{d_{31}^{(k)}}{s_{11}^{E^{(k)}}}E_{3,0}^{(k)}, \quad \sigma_{11,1}^{(k)} = \frac{1}{s^{(k)}}\kappa, \quad D_3^{(k)} = D_{3,0}^{(k)} = \varepsilon_{33}^{T^{(k)}}E_{3,0}^{(k)} + d_{31}^{(k)}\sigma_{11,0}^{(k)}
\end{aligned} \tag{3.11}$$

The stresses in the elastic layer can be calculated from the formula

$$\sigma_{11,1}^{(k)} = E\kappa \tag{3.12}$$

For a piezoelectric layer without electrodes we have

$$\sigma_{11,1}^{(k)} = \frac{1}{s^{(k)}}\kappa, \quad E_{3,1}^{(k)} = -\frac{d_{31}^{(k)}}{\varepsilon_{33}^{T^{(k)}}}\sigma_{11,1}^{(k)} \tag{3.13}$$

Since the stress σ_{11} in each layer varies linearly with x_3, for the stresses σ_{13} and σ_{33} in the bending problem we take the following laws of variation of the stresses over the thickness of the bar

$$\sigma_{13} = \sigma_{13,0} + x_3^2\sigma_{13,2}, \quad \sigma_{33} = x_3\sigma_{33,1} + x_3^3\sigma_{32,3} \tag{3.14}$$

We can obtain approximate values of the stresses σ_{13} and σ_{33}, satisfying conditions (3.10) on the faces of the bar and the equations of motion, using the formula for the shearing force

$$N = -2h\sigma_{13,0} - 2h^3\sigma_{13,2}/3$$

As a result we obtain

$$\begin{aligned}
\sigma_{13} &= -\frac{3}{4h^3}(h^2 - x_3^2)N - \frac{1}{4h^2}(h^2 - 3x_3^2)(q_1^+ - q_1^-) \\
\sigma_{33} &= \frac{x_3}{2h}(q_3^+ - q_3^-) + \frac{x_3}{4h^3}(h^2 - x_3^2)\frac{\partial N}{\partial x_1} + h\frac{\partial}{\partial x_1}(q_1^+ - q_1^-)
\end{aligned} \tag{3.15}$$

Formulae (3.15) define the stresses σ_{13} and σ_{33} as continuous functions, and hence the conditions for the stresses on the contact surfaces of layers to be equal are satisfied automatically. If necessary the stresses σ_{13} and σ_{33} can be found in a higher approximation. To do this the equations of motion must be integrated for each layer separately (they differ from Eq. (3.8) only in that $\partial^2 u/\partial t^2 = 0$ in the bending problem), and then the conditions on the faces of each layer must be satisfied. As a result we obtain

$$\sigma_{13}^{(N)} = (h-x_3)\frac{\partial\sigma_{11,0}^{(N)}}{\partial x_1} + \frac{1}{2}(h^2-x_3^2)\frac{\partial\sigma_{11,1}^{(N)}}{\partial x_1} + \frac{1}{2}(q_1^+ - q_1^-)$$

$$\sigma_{13}^{(k-1)} = (z_{k-1}-x_3)\frac{\partial\sigma_{11,0}^{(k-1)}}{\partial x_1} + \frac{1}{2}(z_{k-1}^2-x_3^2)\frac{\partial\sigma_{11,1}^{(k-1)}}{\partial x_1} + \sigma_{13}^{(k)}\Big|_{x_3=z_{k-1}}, \quad k = N, N-1, \dots, 2$$

$$\sigma_{33}^{(N)} = \left(\frac{h^2}{2}+\frac{x_3^2}{2}-hx_3\right)\frac{\partial^2\sigma_{11,0}^{(N)}}{\partial x_1^2} + \frac{1}{2}\left(\frac{2}{3}h^3+\frac{x_3^3}{3}-h^2x_3\right)\frac{\partial^2\sigma_{11,1}^{(N)}}{\partial x_1^2} - (x_3-h)\rho_N\frac{\partial^2 w}{\partial t^2} -$$

$$-\frac{1}{2}(x_3-h)\frac{\partial}{\partial x_1}(q_1^+-q_1^-) + \frac{1}{2}(q_3^+-q_3^-) \tag{3.16}$$

$$\sigma_{33}^{(k-1)} = \left(\frac{z_{k-1}^2}{2}+\frac{x_3^2}{2}-z_{k-1}x_3\right)\frac{\partial^2\sigma_{11,0}^{(k-1)}}{\partial x_1^2} + \frac{1}{2}\left(\frac{2}{3}z_{k-1}^3+\frac{x_3^3}{3}-z_{k-1}^2x_3\right)\frac{\partial^2\sigma_{11,1}^{(k-1)}}{\partial x_1^2} -$$

$$-\rho_{k-1}(x_3-z_{k-1})\frac{\partial^2 w}{\partial t^2} + \sigma_{33}^{(k)}\Big|_{x_3=z_{k-1}}, \quad k = N, N-1 \dots, 3$$

$$\sigma_{33}^{(1)} = \frac{x_3}{6}(x_3^2-h_1^2)\frac{\partial^2\sigma_{11,1}^{(1)}}{\partial x_1^2} + \frac{x_3}{h_1}\sigma_{33}^{(2)}\Big|_{x_3=h_1}$$

To derive the last relation we use the formula

$$\sigma_{33}^{(1)}\Big|_{x_3=h_1} = \sigma_{33}^{(2)}\Big|_{x_3=h_1}, \quad 3\sigma_{33,3}^{(k)} = -\frac{\partial\sigma_{31,2}^{(k)}}{\partial x_1}, \quad 2\sigma_{13,2}^{(k)} = -\frac{\partial\sigma_{11,1}^{(k)}}{\partial x_1}$$

For elastic layers we have $\sigma_{11,0}^{(k)} = 0$.

4. The electromechanical coupling coefficient

Electroelastic elements are used as energy convertors, hence the electromechanical coupling coefficient (EMCC) is the most important characteristic of their performance.[6–9] It was shown in Refs 6–7 that to calculate the EMCC k_e one can use an energy formula, which has the form

$$k_e = \sqrt{\frac{U^d - U^{sh}}{U^d}}$$

$$U^d = \int_v (\sigma_{ij}^d e_{ij} + E_i^d D_i^d)dv, \quad U^{sh} = \int_v (\sigma_{ij}^{sh} e_{ij} + E_i^{sh} D_i^{sh})dv \tag{4.1}$$

where U^d is the internal energy of the electroelastic body when its electrodes are open-circuit, U^{sh} is the internal energy of the same electroelastic body with short-circuit electrodes and v is the volume of the electroelastic body. In order to find U^d and U^{sh}, we must first solve the initial problem, then two additional problems, one of them for the case of open-circuit electrodes and another for short-circuit electrodes, where, when solving these additional problems, the strains should be considered as known quantities found from the solution of the initial problem.

In order to calculate the internal energy of the electroelastic body with open-circuit electrodes the no–current condition in the open electric circuit is used, namely,

$$I^d = \int_\Omega \frac{dD_3^d}{dt} d\Omega = 0 \tag{4.2}$$

where $d\Omega$ is an element of the surface of one of the electrodes.

The electric potential on the short-circuit electrodes of the piezoelectric layer equals zero

$$\varphi^{sh} = 0 \tag{4.3}$$

To calculated the EMCC the simple Mason formula[9]

$$k_d = \sqrt{\frac{\omega_a^2 - \omega_r^s}{\omega_a^2}} \tag{4.4}$$

is often used, where ω_r is the resonance frequency and ω_a is the corresponding antiresonance frequency.

Formula (4.4) is suitable both for calculation and for processing experimental results, but it only enables one to find the EMCC near resonance. The complicated and time consuming formula (4.1) is a universal formula; using it the EMCC can be calculated for any structure both in statics and dynamics.

5. The vibrations of a three-layer bar

As an example we will consider the harmonic vibrations of a three-layer bar of length l and width g. The upper and lower piezoelectric layers, made of PZT-5 piezoceramics are arranged symmetricly about the middle elastic layer. The thickness of the elastic layer is $2h_1$ and the thickness of each piezoelectric layer is h_2. One edge of the bar is rigidly clamped and another one is free.

5.1. The plane problem (longitudinal vibrations)

It is assumed that only an electrical load exciting longitudinal vibrations acts on the bar:

$$V^{(2)} = V^{(-2)} = V_p \tag{5.1}$$

In the case considered the system of equations of the plane problem (2.16) can be transformed to the following resolvent

$$\frac{d^2u}{d\xi^2} + \lambda_1^2 u = 0 \tag{5.2}$$

where

$$\lambda_1^2 = \frac{2h\rho\omega^2 l^2}{A}, \ \xi = \frac{x_1}{l}, \ A = 2h_1 E + 2h_2 \frac{1}{s_{11}^E}, \ \rho = \frac{1}{h}(\rho_1 h_1 + \rho_2 h_2), \ h = h_1 + h_2 \tag{5.3}$$

the quantities ρ_1 and ρ_2 are the density of the materials of the elastic and piezoelectric layers respectively.

The solution of Eq. (5.2) has the form

$$u = c_1 \sin\lambda_1\xi + c_2 \cos\lambda_1\xi \tag{5.4}$$

From the conditions at the ends of the bar

$$u|_{\xi=0} = 0, \quad T|_{\xi=1} = 0$$

we determine the constants of integration

$$c_1 = -\frac{2V_p}{h_2}\frac{d_{31}l}{\chi\lambda_1\cos\lambda_1}, \quad c_2 = 0; \quad \chi = 1 + Es_{11}^E\frac{h_1}{h_2} \tag{5.5}$$

The displacement and strain of the bar are found from the formulae

$$u = -\frac{2V_p}{h_2}\frac{d_{31}l\sin\lambda_1\xi}{\chi\lambda_1\cos\lambda_1}, \quad \varepsilon = -\frac{2V_p}{h_2}\frac{d_{31}\cos\lambda_1\xi}{\chi\cos\lambda_1} \tag{5.6}$$

We have for the stresses and electrical quantities in the piezoelectric layers

$$\begin{aligned} &\sigma_{11,0} = \frac{1}{s_{11}^E}\varepsilon - \frac{d_{31}}{s_{11}^E}E_{3,0}, \quad \varphi = 2\frac{V_p}{h_2}\left(x_3 - h_1 - \frac{h_2}{2}\right), \quad E_3 = E_{3,0} = -2\frac{V_p}{h_2} \\ &D_{3,0} = \varepsilon_{33}^T E_{3,0} + d_{31}\sigma_{11,0}, \quad I = i4\omega l\varepsilon_{33}^T k_{31}^2\frac{V_p}{h_2}\left(K^{-1} + \frac{\mathrm{tg}\lambda_1}{\chi\lambda_1}\right), \quad K = \frac{k_{31}^2}{1 - k_{31}^2} \end{aligned} \tag{5.7}$$

and for the stresses in the elastic layer

$$\sigma_{11,0} = E\varepsilon \tag{5.8}$$

The stresses σ_{33} and σ_{13} are determined from formulae (3.9), in which the mechanical surface load must be put equal to zero.

Calculating the EMCC k_e from formula (4.1) we obtain

$$U^d = \int_{v_2}(\sigma_{11,0}^d\varepsilon + E_{3,0}^d D_{3,0}^d)dv_2 + \int_{v_1}\sigma_{11,0}\varepsilon dv_1 \tag{5.9}$$

$$U^{sh} = \int_{v_2}(\sigma_{11,0}^{sh}\varepsilon + E_{3,0}^{sh} D_{3,0}^{sh})dv_2 + \int_{v_1}\sigma_{11,0}\varepsilon dv_1 \tag{5.10}$$

Here v_1 and v_2 are the areas occupied by the elastic and piezoelectric layers respectively.

Using the no-current condition in the open circuit

$$I^d = \int_{\Omega}\frac{dD_3^d}{dt}d\Omega = 0 \tag{5.11}$$

where

$$D_3^d = \frac{d_{31}}{s_{11}^E}\varepsilon + \varepsilon_{33}^T(1 - k_{31}^2)E_{3,0}^d = -2\frac{\varepsilon_{33}^T}{h_1}(1 - k_{31}^2)\left(V^d + K\frac{\cos\lambda_1\xi}{\chi\cos\lambda_1}V_p\right) \tag{5.12}$$

we find

$$V^d = -K\frac{\mathrm{tg}\lambda_1}{\chi\lambda_1}V_p, \quad E_{3,0}^d = -2\frac{V^d}{h_2} = 2K\frac{\mathrm{tg}\lambda_1}{\chi\lambda_1}\frac{V_p}{h_2} \tag{5.13}$$

On the short-circuit electrodes the electric potential and $E_{3,0}^{sh}$ are equal to zero:

$$V^{sh} = 0, \quad E_{3,0}^{sh} = 0 \tag{5.14}$$

The stress $\sigma_{11,0}$ in the elastic layer is determined from formula (5.8).

If the formulae obtained are taken into account, the stresses in the piezoelectric layers can be written as follows:

$$\sigma_{11,0}^d = \frac{1}{s_{11}^E}\varepsilon + \frac{d_{31}}{s_{11}^E}\frac{2V^d}{h_2}, \quad \sigma_{11,0}^{sh} = \frac{1}{s_{11}^E}\varepsilon \tag{5.15}$$

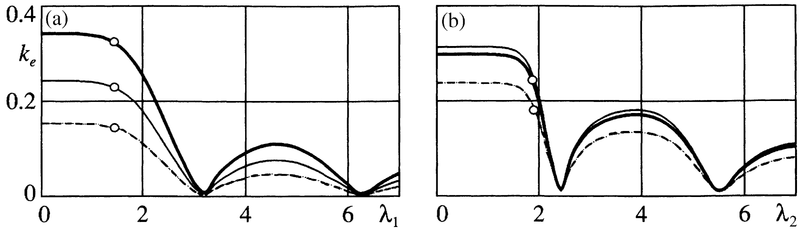

Fig. 2.

We substitute expressions (5.12)–(5.15) into relations (5.9), (5.10) and perform the integration

$$U^d = \int_{\nu_2}\left(\frac{1}{s_{11}^E}\varepsilon^2 + \varepsilon_{33}^T(1-k_{31}^2)(E_{3,0}^d)^2\right)d\nu_2 + \int_{\nu_1} E\varepsilon^2 d\nu_1$$

$$U^d = P_1 k_{31}^2(4\sin^2\lambda_1 + K^{-1}\chi\lambda_1(2\lambda_1 + \sin 2\lambda_1))$$

$$U^{sh} = \int_{\nu_2}\sigma_{11,0}^{sh}\varepsilon d\nu_2 + \int_{\nu_1}\sigma_{11,0}\varepsilon d\nu_1 = \int_{\nu_2}\frac{1}{s_{11}^E}\varepsilon^2 d\nu_2 + \int_{\nu_1}E\varepsilon^2 d\nu_1 \tag{5.16}$$

$$U^d - U^{sh} = \varepsilon_{33}^T(1-k_{31}^T)\int_{\nu_1}(E_3^d)^2 d\nu_1 = 4P_1 k_{31}^2 \sin^2\lambda_1$$

where

$$P_1 = K\varepsilon_{33}^T\frac{\nu_1}{\lambda_1\chi^2\cos^2\lambda_1}\left(\frac{V_p}{h_2}\right)^2, \quad \frac{\nu_1}{\nu_2} = \frac{h_1}{h_2}$$

and we then find the EMCC from formula (4.1)

$$k_e^2 = \frac{4K\sin^2\lambda_1}{4K\sin^2\lambda_1 + \chi\lambda_1(2\lambda_1 + \sin 2\lambda_1)} \tag{5.17}$$

Graphs of k_e against the dimensionless frequency parameter λ_1 for different values h_1 and h_2 are shown in Fig. 2a (the thick curve corresponds to $h_1 = 0$ and $h_2 = 1 \times 10^{-3}$ m; the thin curve corresponds to $h_1 = h_2 = 5 \times 10^{-4}$ m; the dashed curve corresponds to $h_1 = 8 \times 10^{-4}$ m and $h_2 = 2 \times 10^{-4}$ m).

Table 2 gives the values of the first (subscript 1) and second (subscript 2) dimensionless (λ) and dimensional (ω, kHz) resonance (subscript r) and antiresonance (subscript a) frequencies and the values of the EMCC k_d calculated using Mason's formula (4.4) (they are shown in Fig. 2 by the light circles). As in all the other problems considered[6,7] the EMCC values determined by formulae (4.1) and (4.4) agree.

Table 2

Frequencies and EMCC	$h_1 = 0$, $h_2 = 10^{-3}$ m	$h_1 = h_2 = 5 \times 10^{-4}$ m	$h_1 = 8 \times 10^{-4}$ m, $h_2 = 2 \times 10^{-4}$ m
λ_{r1} (ω_r, kHz)	1.571 (35.22 kHz)	1.571 (44.49 kHz)	1.571 (54.21 kHz)
λ_{a1} (ω_a, kHz)	1.651 (37.01 kHz)	1.610 (45.58 kHz)	1.586 (54.73 kHz)
k_{d1}	0.307	0.218	0.137
λ_{r2} (ω_r, kHz)	4.712 (105.7 kHz)	4.712 (133.5 kHz)	4.712 (162.6 kHz)
λ_{a2} (ω_a, kHz)	4.740 (106.3 kHz)	4.726 (133.8 kHz)	4.717 (162.8 kHz)
k_{d2}	0.108	0.075	0.046

5.2. The bending problem

For the complete system of equations of the bending problem

$$N = \frac{dG}{dx_1}, \quad \frac{dN}{dx_1} = -2h\rho\omega^2 w, \quad V_b = V^{(2)} = -V^{(-2)}$$

$$G = M\kappa - \frac{d_{31}}{s_{11}^E}(h^2 - h_1^2)\frac{2V_b}{h_2}, \quad M = -\frac{2}{3}Eh_1^3 - \frac{2}{3s_{11}^E}\left((h^3 - h_1^3) + K\frac{h_2^3}{4}\right)$$

we have the resolvent

$$\frac{d^2 w}{d\xi^4} + \lambda_2^4 w = 0, \quad \lambda_2^4 = -\frac{2h\rho\omega^2 l^4}{M}, \quad \xi l = x_1$$

and its solution

$$w = c_1 \operatorname{ch}\lambda_2\xi + c_2 \operatorname{sh}\lambda_2\xi + c_3 \cos\lambda_2\xi + c_4 \sin\lambda_2\xi$$

Satisfying the conditions of rigid clamping at the edge $x_1 = 0$

$$w|_{x_1 = 0} = 0, \quad \left.\frac{dw}{dx_1}\right|_{x_1 = 0} = 0$$

and the conditions at the free edge $x_1 = l$

$$G|_{x_1 = l} = 0, \quad N|_{x_1 = l} = 0$$

we obtain the constants of integration

$$\begin{aligned} c_1 &= -c_3 = \frac{2V_b}{h_2}\frac{d_{31}l^2(h^2 - h_1^2)}{s_{11}^E M\lambda_2^2\delta}(\operatorname{ch}\lambda_2 + \cos\lambda_2) \\ c_2 &= -c_4 = \frac{2V_b}{h_2}\frac{d_{31}l^2(h^2 - h_1^2)}{s_{11}^E M\lambda_2^2\delta}(\operatorname{sh}\lambda_2 - \sin\lambda_2) \\ \delta &= 2(\operatorname{ch}\lambda_2\cos\lambda_2 + 1) \end{aligned} \tag{5.18}$$

and the deflection

$$w \mp c_1(\operatorname{ch}\lambda_2\xi - \cos\lambda_2\xi) + c_2(\operatorname{sh}\lambda_2\xi - \sin\lambda_2\xi) \tag{5.19}$$

We will write the formulae for some of the required quantities

$$\kappa = \frac{\lambda_2^2}{l^2}[c_1(\mathrm{ch}\lambda_2\xi + \cos\lambda_2\xi) + c_2(\mathrm{sh}\lambda_2\xi + \sin\lambda_2\xi)]$$

$$\sigma_{11,0} = -\frac{d_{31}}{s_{11}^E}E_{3,0}, \quad \varphi_{,2} = \frac{d_{31}}{2\varepsilon_{33}^T}\sigma_{11,1} = \frac{1}{2d_{31}}K\kappa$$

$$E_{3,0} = -\frac{2V_b}{h_2} + \frac{h+h_1}{2d_{31}}K\kappa, \quad D_{3,0} = \varepsilon_{33}^T E_{3,0} + d_{31}\sigma_{11,0} = \varepsilon_{33}^T(1-k_{31}^2)E_{3,0} \tag{5.20}$$

$$I = \int_\Omega \frac{dD_3}{dt}d\Omega = -i\omega g\varepsilon_{33}^T(1-k_{31}^2)\frac{2V_b}{h_2}\left[-1 + \frac{h_2}{2V_b}\frac{h+h_1}{2d_{31}}K\frac{dw}{l^2 d\xi}\bigg|_{\xi=1}\right]$$

$$-i\omega gl\varepsilon_{33}^T(1-k_{31}^2)\frac{2V_b}{h_2}\left[-1 + K\frac{(h+h_1)^2 h_2}{s_{11}^E M\lambda_2^2\delta}(\mathrm{ch}\lambda_2\sin\lambda_2 + \mathrm{sh}\lambda_2\cos\lambda_2)\right]$$

The stresses in the layers are given by formulae (3.11)–(3.16). In order to calculate the EMCC we will use formulae (4.1) and (4.4). We find V_b^d by satisfying condition (4.2). We obtain

$$\sigma_{11,0}^d = -\frac{d_{31}}{s_{11}^E}E_{3,0}^d, \quad \sigma_{11,1}^d = \sigma_{11,1}^{sh} = \frac{\varepsilon_{33}^T}{d_{31}^2}K\kappa$$

$$E_{3,0}^d = -\frac{2V_b^d}{h_2} + \frac{h+h_1}{2d_{31}}K\kappa, \quad D_{3,0}^d = \varepsilon_{33}^T(1-k_{31}^2)E_{3,0}^d \tag{5.21}$$

$$V_b^d = V_b K\frac{(h+h_1)(h^2-h_1^2)}{s_{11}^E M\lambda_2^2\delta}(\mathrm{ch}\lambda_2\sin\lambda_2 + \mathrm{sh}\lambda_2\cos\lambda_2)$$

According to condition (4.3)

$$E_{3,0}^{sh} = \frac{h+h_1}{2d_{31}}K\kappa, \quad \sigma_{11,0}^{sh} = -\frac{d_{31}}{s_{11}^E}E_{3,0}^{sh}$$

$$V^{sh} = 0, \quad D_{3,0}^{sh} = \varepsilon_{33}^T(1-k_{31}^2)\frac{h+h_1}{2d_{31}}K\kappa \tag{5.22}$$

Using formulae (5.22) we find

$$U^d = P_2(1+\mu C), \quad U^d - U^{sh} = P_2 \tag{5.23}$$

where

$$P_2 = \frac{2\varepsilon_{33}^T(1-k_{31}^2)(h+h_1)^4 h_2 S^2}{(\lambda_2\delta s_{11}^E M)^2}\left(\frac{2V_b}{h_2}\right)^2, \quad \mu = -\frac{\lambda_2 s_{11}^E M}{2K(h+h_1)^2 h_2}$$

$$C = \lambda_2\left(\frac{\cos\lambda_2 + \mathrm{ch}\lambda_2}{S}\right)^2 + 3\frac{1+\cos\lambda_1\mathrm{ch}\lambda_2}{S}, \quad S = \cos\lambda_2\mathrm{sh}\lambda_2 + \sin\lambda_2\mathrm{ch}\lambda_2$$

After simple rearrangement formula (4.1) takes the form

$$k_e^2 = \frac{1}{1+\mu C} \tag{5.24}$$

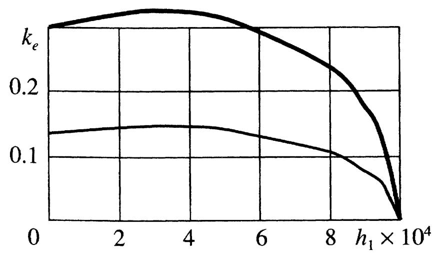

Fig. 3.

Table 3

Frequencies and EMCC	$h_1 = 0$, $h_2 = 10^{-3}$ m	$h_1 = h_2 = 5 \times 10^{-4}$ m	$h_1 = 8 \times 10^{-4}$ m, $h_2 = 2 \times 10^{-4}$ m
λ_{r1} (ω_r, kHz)	1.875 (2.312 kHz)	1.875 (2.813 kHz)	1.875 (3.441 kHz)
λ_{a1} (ω_a, kHz)	1.925 (2.440 kHz)	1.930 (2.974 kHz)	1.907 (3.563 kHz)
k_{d1}	0.229	0.237	0.183
λ_{r2} (ω_r, kHz)	4.694 (14.48 kHz)	4.694 (17.58 kHz)	4.694 (21.53 kHz)
λ_{a2} (ω_a, kHz)	4.735 (14.74 kHz)	4.740 (17.93 kHz)	4.720 (21.77 kHz)
k_{d2}	0.132	0.139	0.105

Graphs of the EMCC k_e against the dimensionless frequency parameter λ_2 for different thicknesses of the layers, identical with those used in Fig. 2a, are presented in Fig. 2b.

Fig. 3 shows the dependence of the EMCC on the thickness of the elastic layer, assuming that the total-thickness of the bar remains constant (the half-thickness $h = 10^{-3}$ m) for the first (the thick curve) and the second (the thin curve) resonance frequencies. Calculations show that the maximum value of the EMCC is achieved when $h_1 = 3.5 \times 10^{-4}$ m, and this value does not depend on the frequency of bending vibrations.

The values of the first and second resonance and antiresonance frequencies and the values of the EMCC calculated from formula (4.4) are given in Table 3, like Table 2.

Here, as in all the other problems considered, the EMCC values found using formulae (4.1) and (4.4) agree.

The dependence of the dimensionless electric potential $\varphi^* = \varphi/V_b$ on the thickness coordinate x_3 is shown in Fig. 4. It can be seen that the electric potential varies quadratically with the thickness, which is in good agreement with a numerical calculation using the three-dimensional theory.[2]

The variation of the dimensionless stresses

$$\sigma_{11*} = \frac{h_2}{2V_b}\frac{s_{11}^E}{d_{31}}\sigma_{11}, \quad \sigma_{31*} = \frac{h_2}{2V_b}\frac{s_{11}^E}{d_{31}}\sigma_{31}$$

with the thickness of the bar in the cross section of the bar $x_1 = l/2$ close to the first resonance ($\lambda_2 = 1.86$) is shown in Fig. 5. It can be seen that the stress σ_{11*} varies linearly with the thickness while σ_{31*} varies quadratically; moreover the stress σ_{11*} is considerably greater than the stress σ_{31*}.

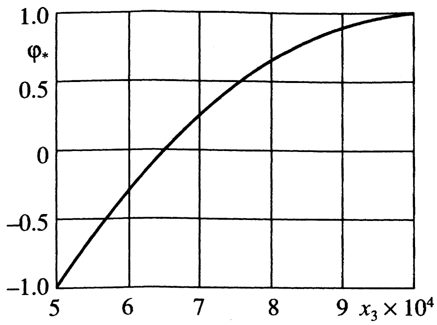

Fig. 4.

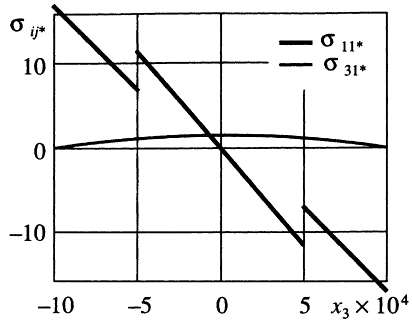

Fig. 5.

5.3. Simultaneous longitudinal and transverse vibrations of the bar

If the bar simultaneously performs forced longitudinal and transverse vibrations, then, as is well-known, the complete problem can be split into a plane problem for the longitudinal vibrations and a bending problem for the transverse vibrations. All the required quantities are determined by simple addition of the quantities found from the solutions of the plane problem and bending problem. The exception is the EMCC which cannot be found by simple addition.

Substitution of the total stresses, strains and electrical quantities into the formulae for the energy (4.1) leads to the following formulae for energy and the EMCC

$$U^d = \varsigma_b^2 U_b^d + \varsigma_p^2 U_p^d, \quad U^{sh} = \varsigma_b^2 U_b^{sh} + \varsigma_p^2 U_p^{sh},$$
$$k_e^2 = \frac{(U_b^d - U_b^{sh}) + \nu^2 (U_p^d - U_p^{sh})}{U_b^d + \nu^2 U_p^d} \quad \varsigma_b = \frac{2V_b}{h_2}, \quad \varsigma_p = \frac{2V_p}{h_2}, \quad \nu = \frac{V_p}{V_b} \tag{5.25}$$

where U_b^d, U_b^{sh}, U_p^d and U_p^{sh} are the values calculated for unit electrical load exciting transverse vibrations (with subscript b) and longitudinal vibrations (with subscript p) of the bar. U_b^d and U_b^{sh} are given by formulae (5.23) and U_p^d and U_p^{sh} are given by formulae (5.16).

The results of a calculation of the EMCC as a function of the dimensionless frequency parameter λ_1 for a three-layer bar with layer thicknesses $h_1 = h_2 = 5 \times 10^{-4}$ m and different values of ν are presented in Fig. 6. If $\nu = 50$, the graph of the EMCC coincides with the corresponding graph of the EMCC for longitudinal vibrations shown in Fig. 2b; if $\nu = 3$, the plot of the EMCC coincides with the plot of the EMCC for transverse vibrations (Fig. 4). When $\nu = 15$ both the longitudinal and transverse vibrations may affect the values of the EMCC.

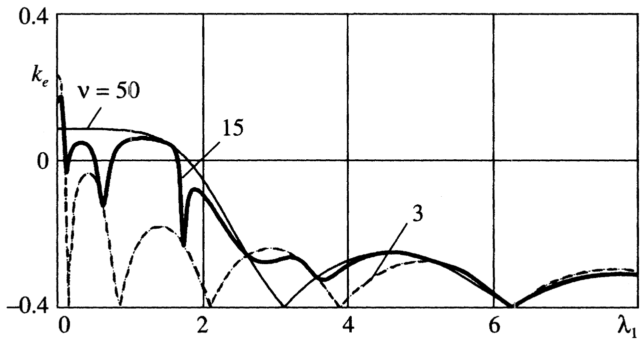

Fig. 6.

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